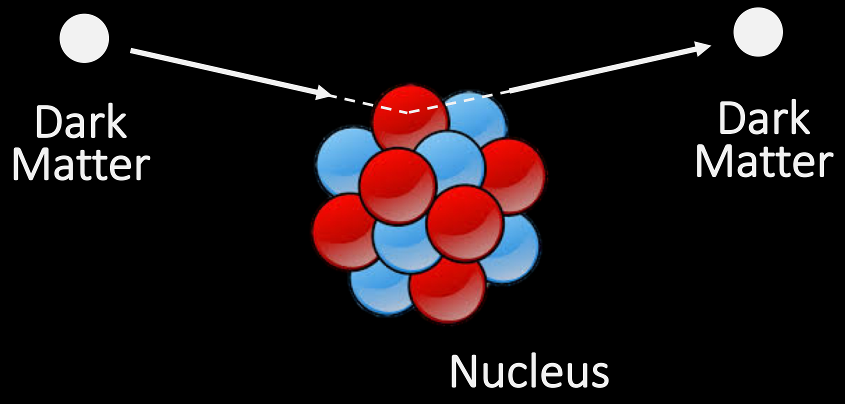
Impact of Nuclear Structure on Nuclear Responses to WIMP Elastic Scattering

2023 CDM Annual Workshop

Australian National University

ARC Centre of Excellence for Dark Matter

Raghda Abdel Khaleq



My Gang

The team

Prof. Cedric Simenel

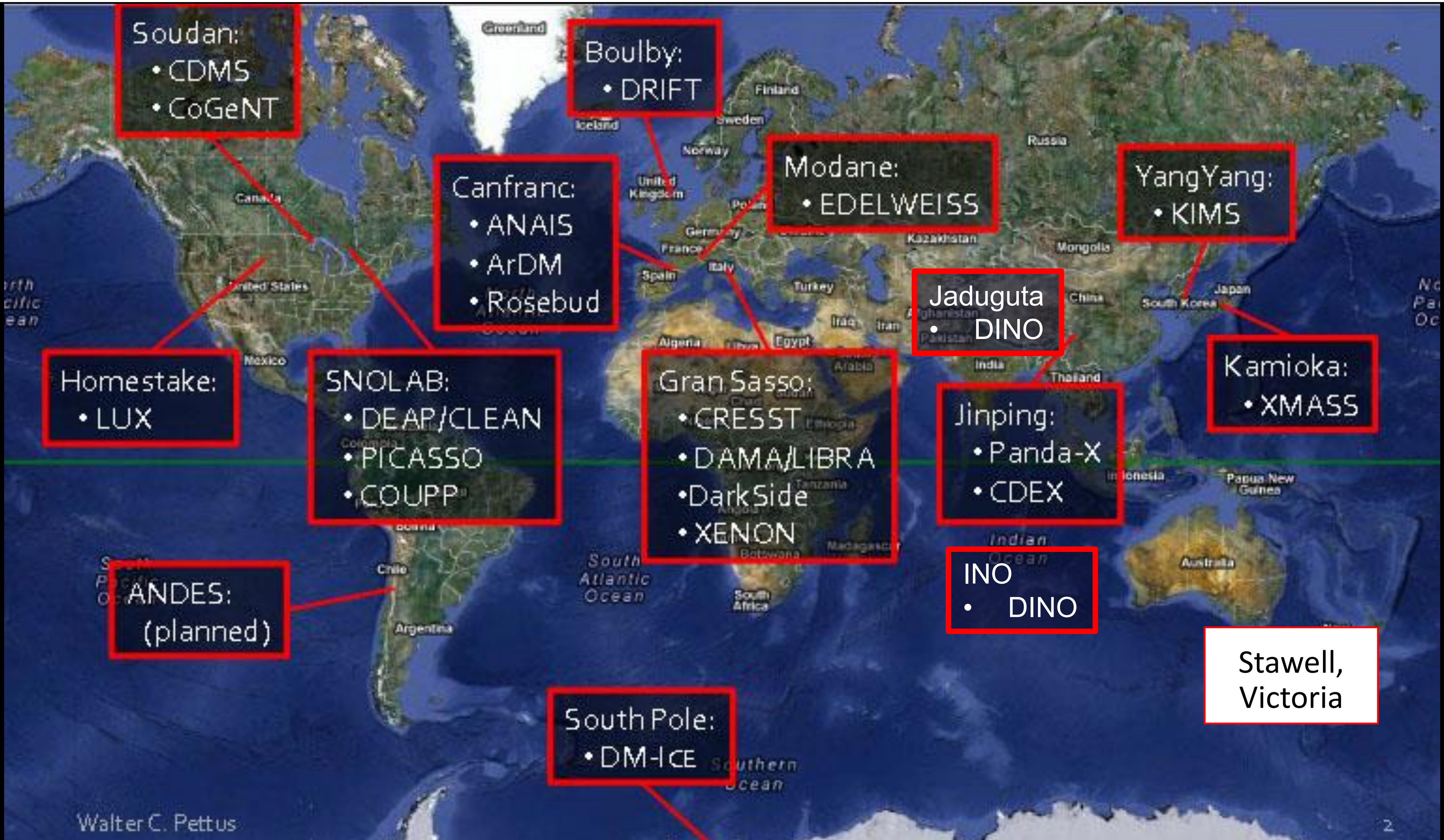
Prof. Andrew Stuchbery

Dr Giorgio Busoni

Cultural Snippet - Palestine

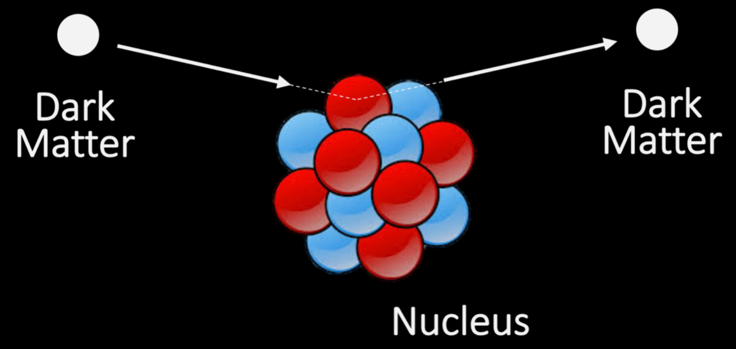
Tatreez (embroidery)

Direct Detection Searches



DM-Nucleus Elastic Scattering

Building blocks of dark matter-nucleus elastic scattering theory



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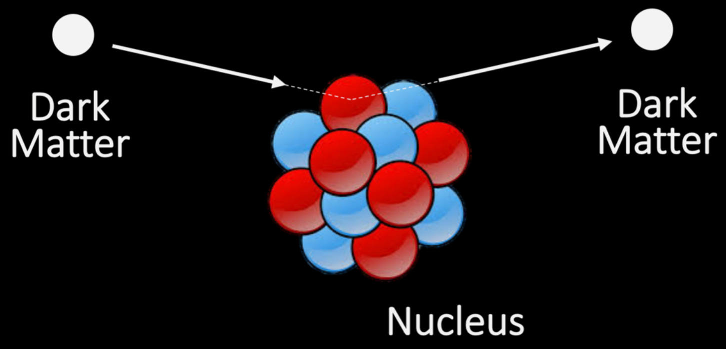
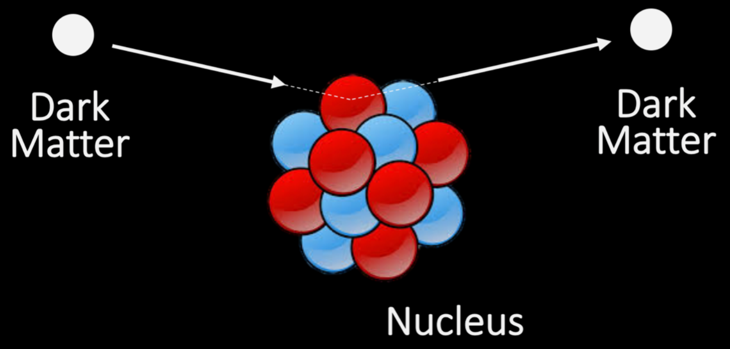
Building blocks of dark matter-nucleus elastic scattering theory



$$\underbrace{\frac{dR}{dE_R}}_{\text{Differential scattering (interaction) rate}} \propto \int v\, d^3v \sum_{ij} \sum_{N,N'=p,n} \underbrace{f_v(\vec{v})}_{\text{DM Velocity Distribution}} \; \underbrace{R(\vec{v},q)_{ij}^{(N,N')}}_{\text{Particle Physics}} \; \underbrace{F(q)_{ij}^{(N,N')}}_{\text{Nuclear Structure}}$$

DM-Nucleus Elastic Scattering

Experiments use different nuclei as targets



$$\underbrace{\frac{dR}{dE_R}}_{\text{Differential scattering (interaction) rate}} \propto \int v\, d^3v \sum_{ij} \sum_{N,N'=p,n} \underbrace{f_v(\vec{v})}_{\text{DM Velocity Distribution}} \underbrace{R(\vec{v},q)_{ij}^{(N,N')}}_{\text{Particle Physics}} \underbrace{F(q)_{ij}^{(N,N')}}_{\text{Nuclear Structure}}$$

Research Goal

Effect of nuclear structure: a neglected aspect

Investigate the sensitivity of Dark Matter-Nucleus scattering to nuclear structure

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$$\frac{dR}{dE_R} \propto \int v\, d^3v \sum_{ij} \sum_{N,N'=p,n} f_v(\vec{v}) \quad R(\vec{v},q)_{ij}{}^{(N,N')} \quad F(q)_{ij}^{(N,N')}$$

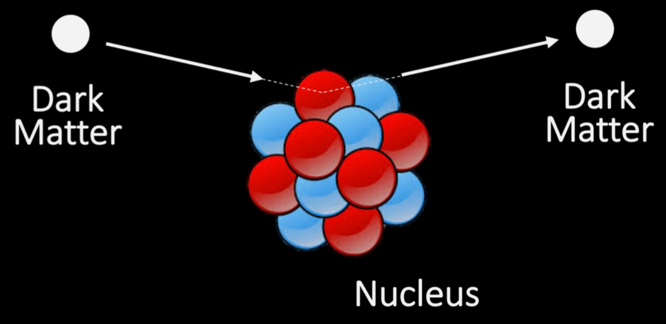
Differential scattering (interaction) rate

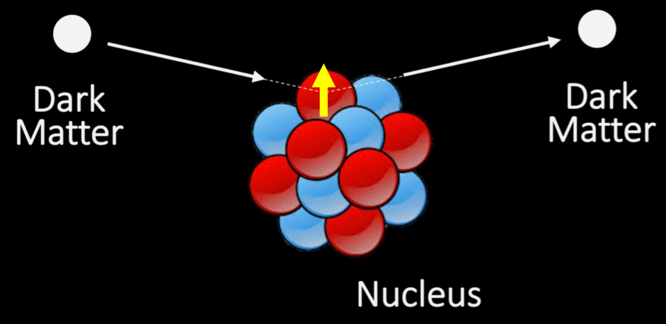
Form Factor - Nuclear Structure

Nuclear Structure– Standard Characterisation

Early models use a simple picture

Spin-independent (SI)	&	Spin-dependent (SD)
Takes into account all nucleons		*Sensitive to nucleon spin*





Nuclear Structure– Standard Characterisation

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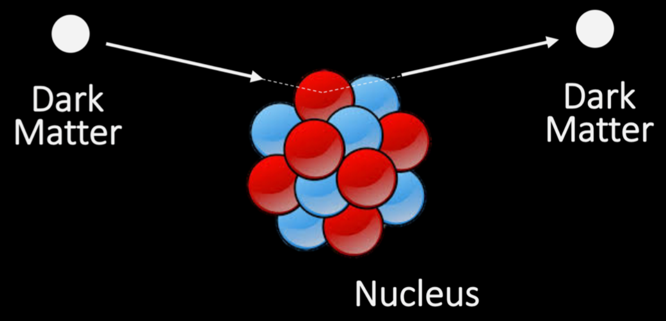
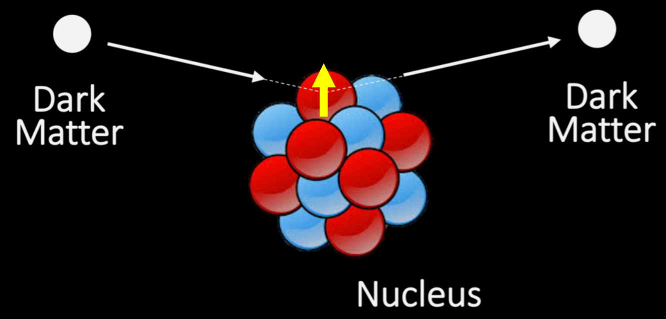




⟹ *Need to consider motions of nucleons in nucleus!*

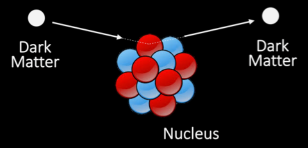
New Interactions From Nucleon Motion

$\boxed{F(q)_{ij}^{(N,N')} \text{ decomposed into}}$

Standard

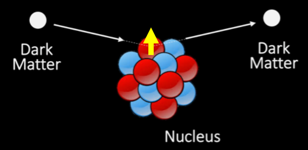
M_{JM}

Spin-independent (SI)



$\Sigma'_{JM}, \Sigma''_{JM}$

Spin-dependent (SD)

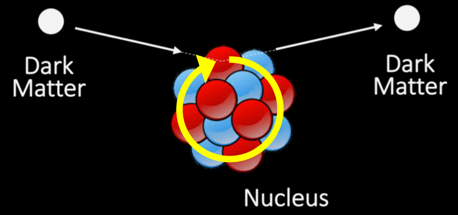


$+$

Additional channels

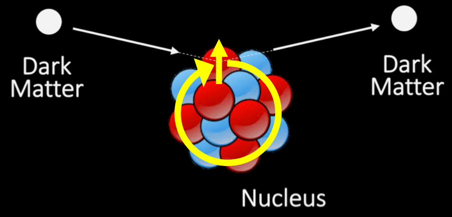
Δ_{JM}

Nucleon orbital angular momentum



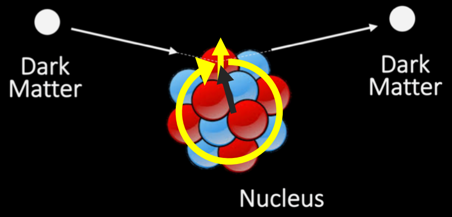
Φ''_{JM}

Spin-orbit structure



$\tilde{\Phi}'_{JM}$

Tensor term



- A. L. Fitzpatrick, W. Haxton, E. Katz, N. Lubbers, and Y. Xu, Journal of Cosmology and Astroparticle Physics (2013), ISSN 14757516.
- N. Anand, A. L. Fitzpatrick, and W. C. Haxton (2013), URL http://arxiv.org/abs/1308.6288

New Interactions From Nucleon Motion

Standard | *Additional channel*

M_{JM} $\Sigma'_{JM}, \Sigma''_{JM}$ $+$ $_{JM}$ $\tilde{\Phi}'_{JM}$

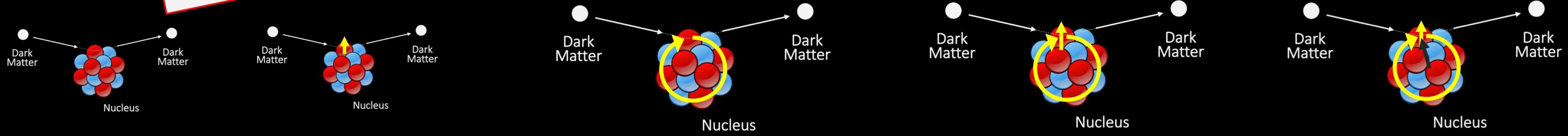
Spin-independen

…rbital angular momentum

Spin-orbit structure

Tensor term

Need to choose specific nuclear model to use



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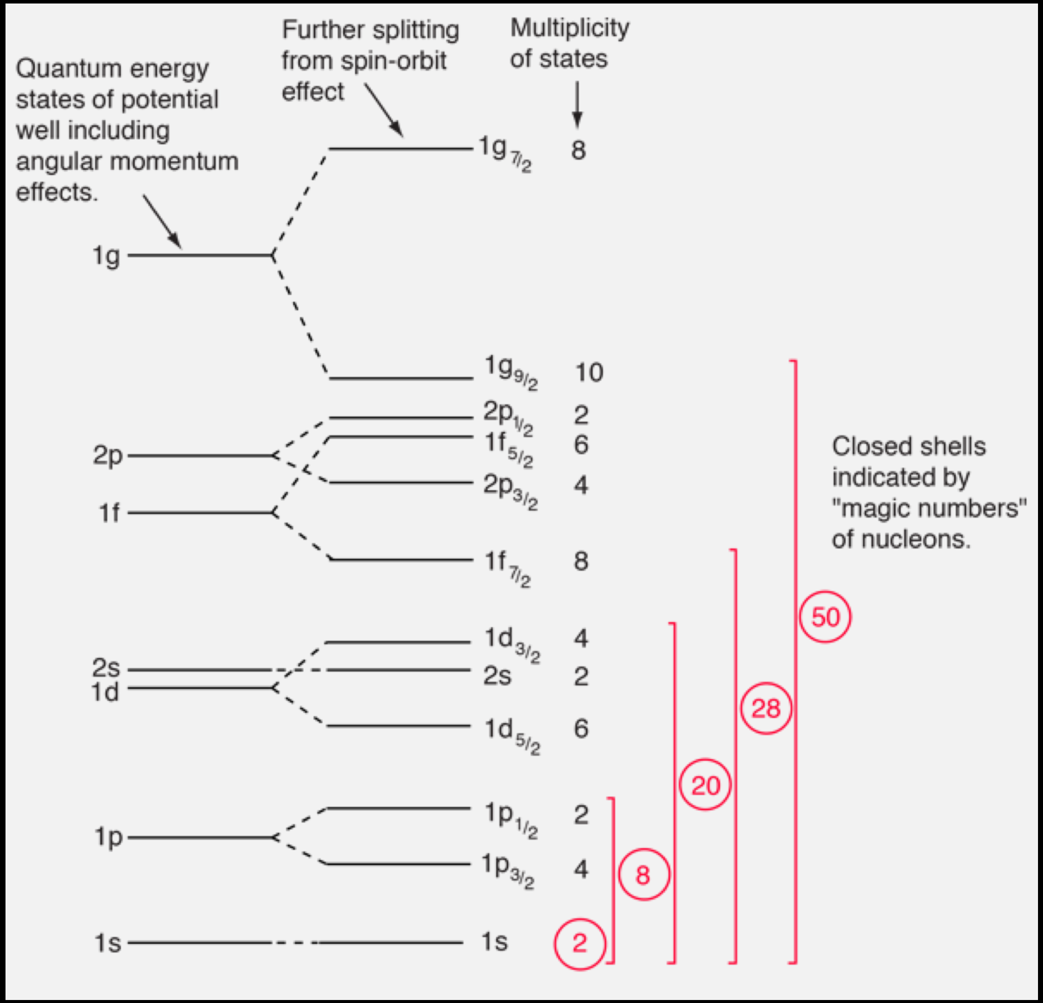
The Setup – Nuclear Shell Model

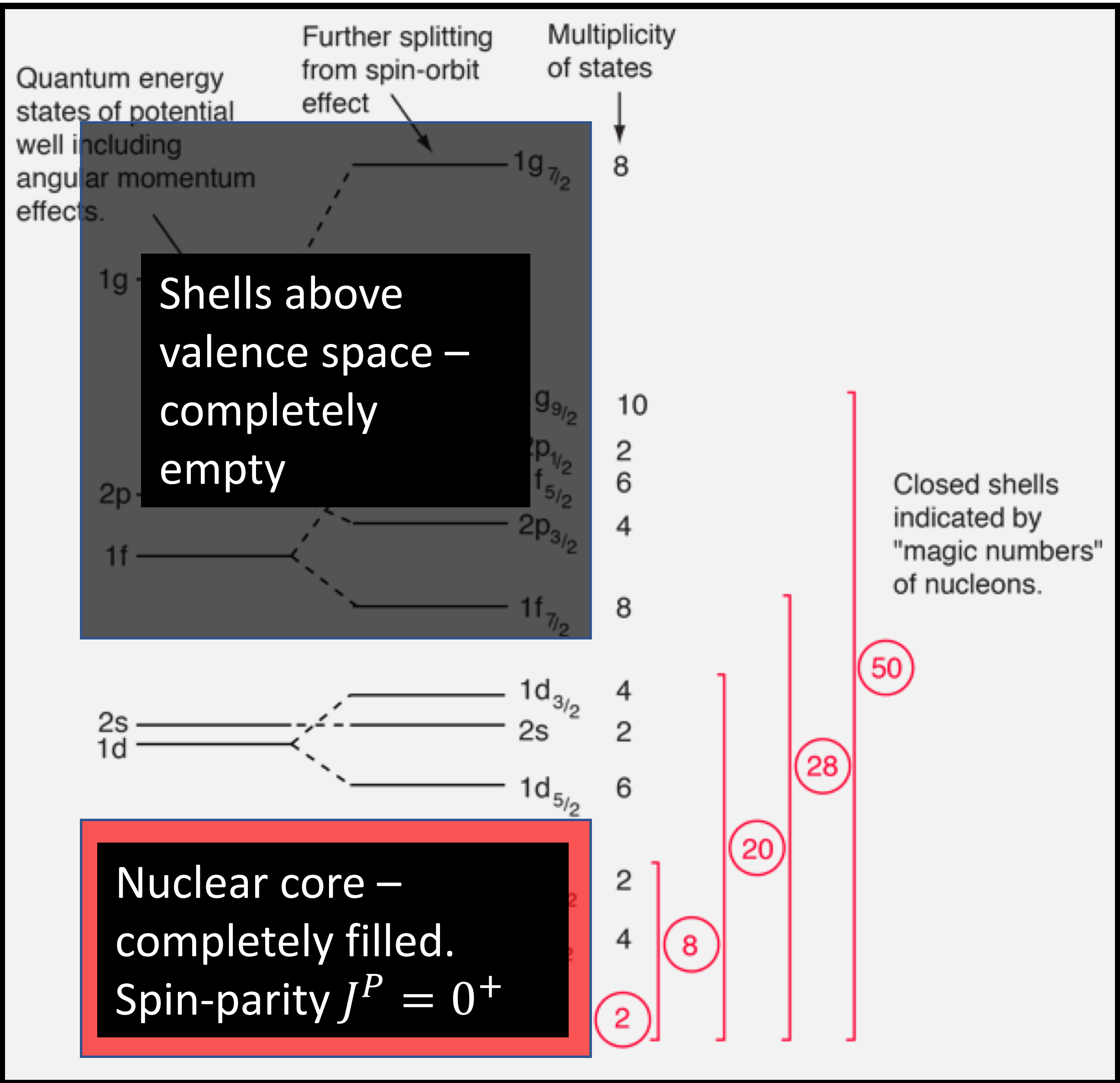
Nucleons in orbits & shells within nucleus, move in effective potential.



The Setup – Nuclear Shell Model

Nucleons in orbits & shells within nucleus, move in effective potential.

Angular momentum, parity and overall nuclear wave function dictated by the valence nucleons – those outside the core.



NuShellX

Used NuShellX (nuclear shell model program) to calculate FF values.

Program Inputs

- Pick valence (model) space
- Pick shell model interaction – each fitted to different nuclear data, hence each might give different FF values

Research Completed

For the nuclei/isotopes

$^{19}F,\ ^{23}Na,\ ^{28,29,30}Si,\ ^{40}Ar,\ ^{70,72,73,74,76}Ge$

$^{127}I,\ ^{128,129,130,131,132,134,136}Xe$

Impact of shell model interactions on nuclear responses to WIMP elastic scattering

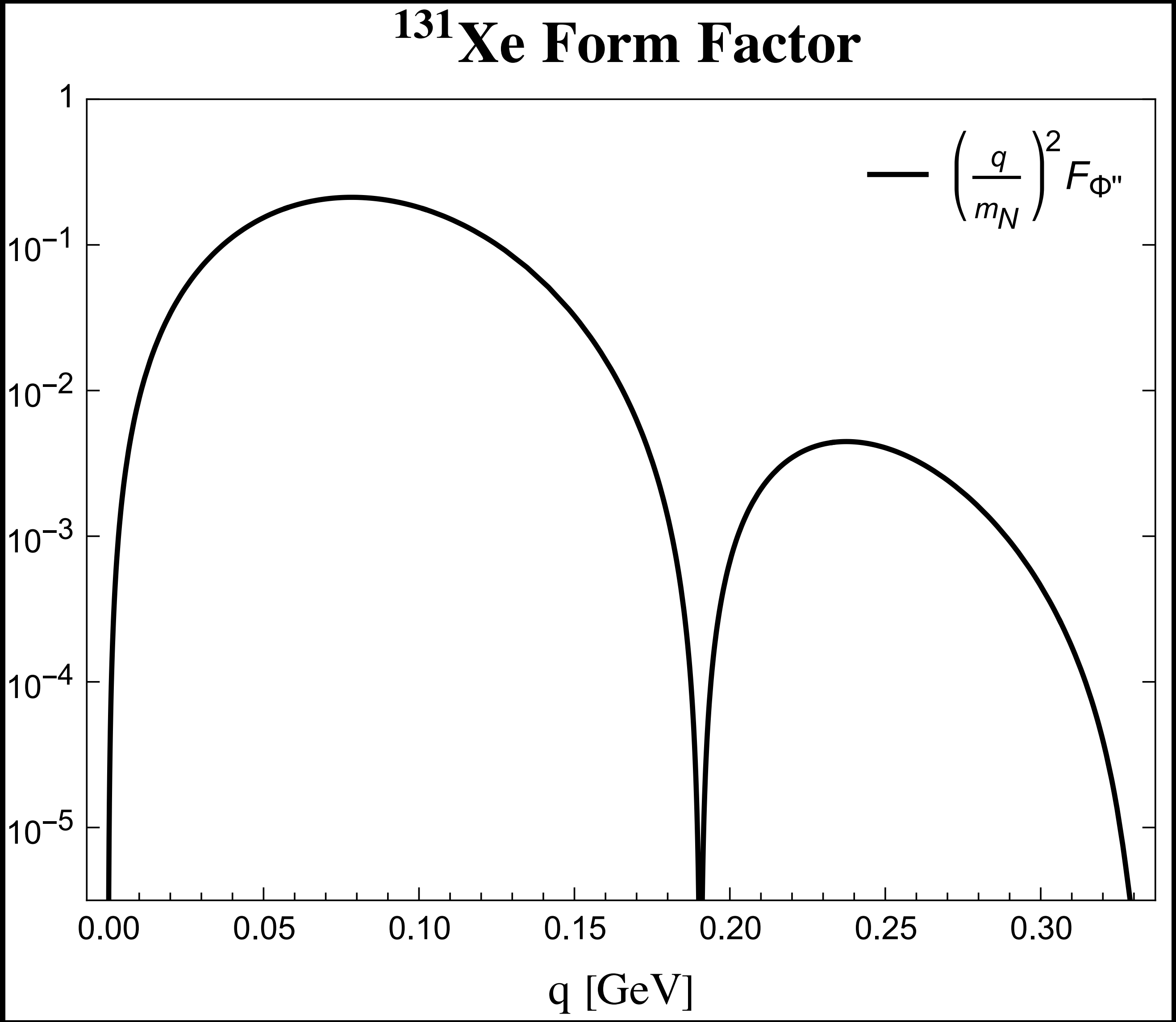
Raghda Abdel Khaleq, Giorgio Busoni, Cedric Simenel, Andrew E. Stuchbery

Background: Nuclear recoil from scattering with weakly interacting massive particles (WIMPs) is a signature searched for in direct detection of dark matter. The underlying WIMP–nucleon interactions could be spin and/or orbital angular momentum

arXiv: 2311.15764

Nuclear Input in Form Factors

- Hold nuclear structure information
- Indicate scattering probability as function of momentum transfer q



Nuclear Input in Form Factors

- Hold nuclear structure information
- Indicate scattering probability as function of momentum transfer q

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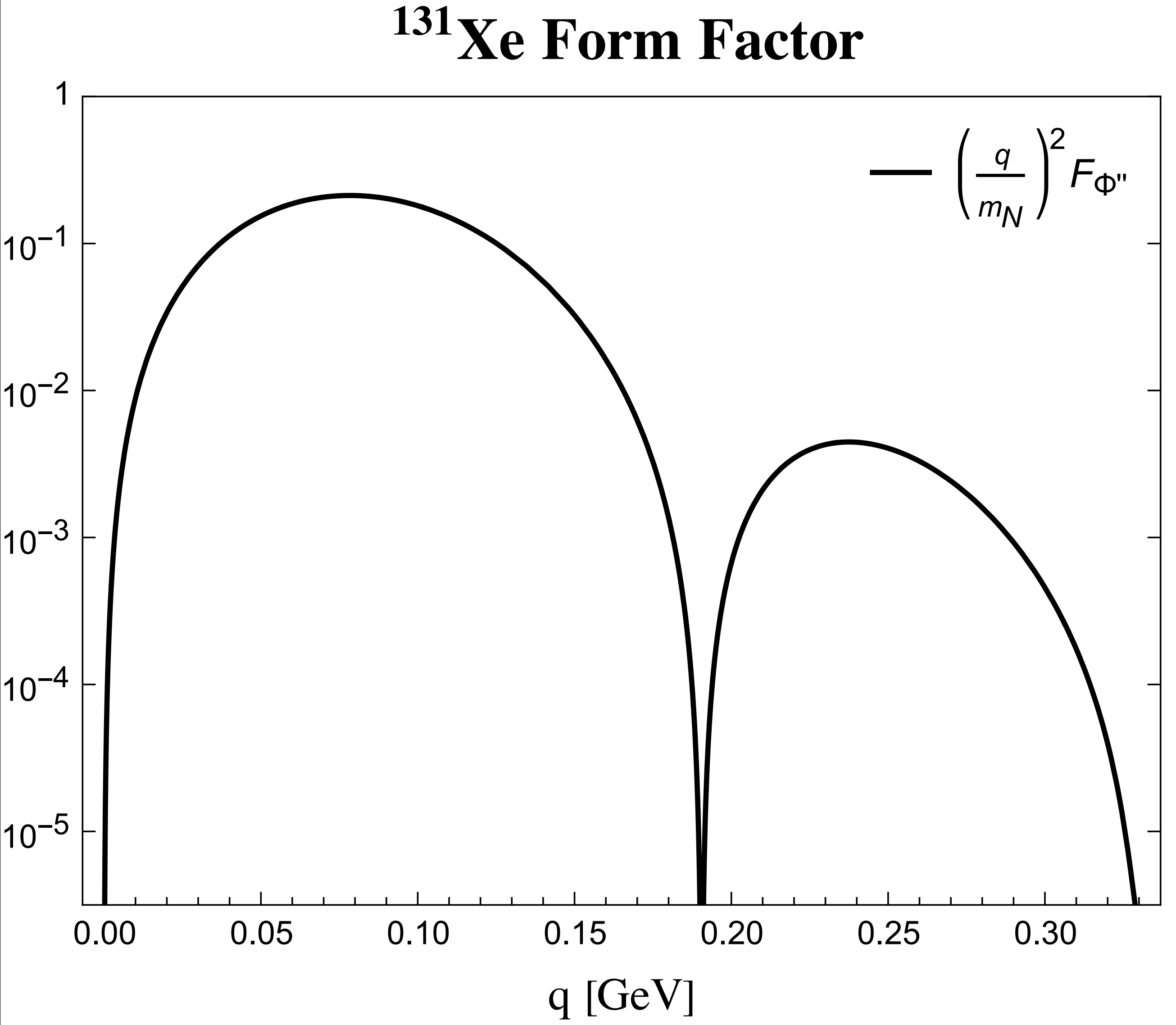
Integrated Form Factor (IFF)

$$\int_0^{100MeV} \frac{qdq}{2} F_X^{(N,N)}(q^2),$$

in units of $(\mathrm{MeV})^2$

where $N = N' = p, n$

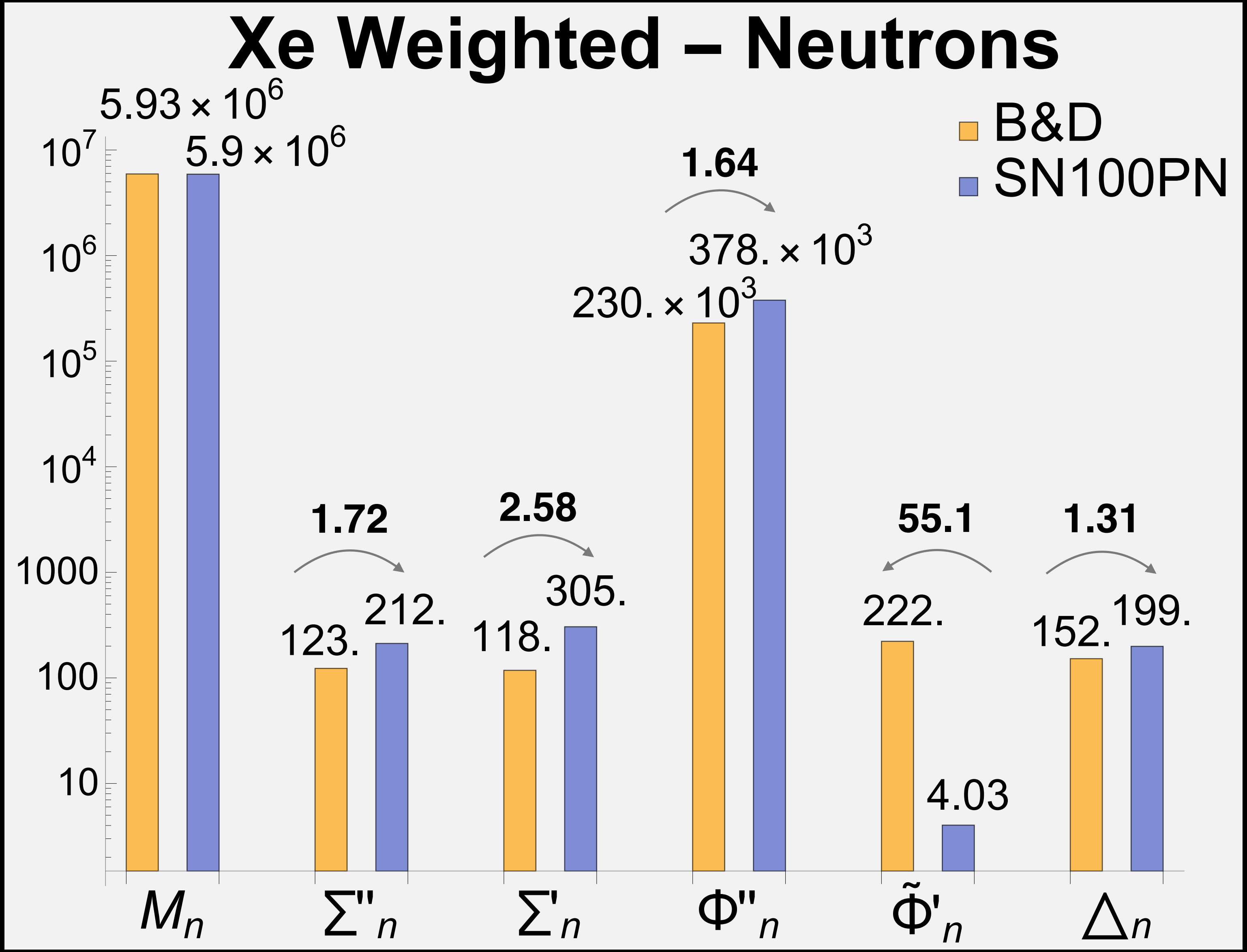


Integrated Form Factor Results

$$\int_0^{100MeV} \frac{qdq}{2} F_X^{(N,N)}(q^2)$$

$^{128,129,130,131,132,134,136}Xe$

Weighted by isotopic abundance



Integrated Form Factor Results

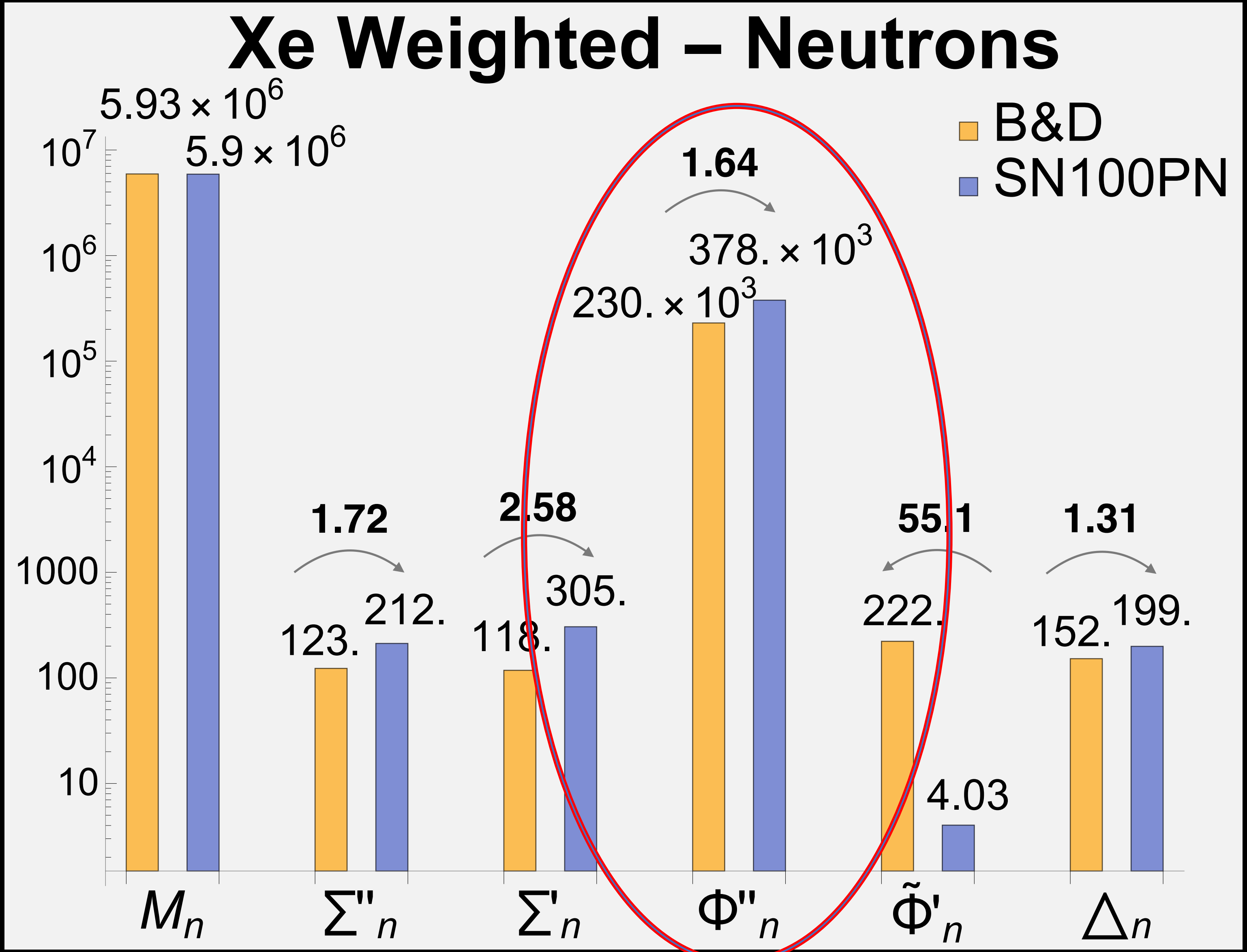
$$\int_0^{100MeV} \frac{qdq}{2} F_X^{(N,N)}(q^2)$$

Φ'' = spin-orbit term

$^{128,129,130,131,132,134,136}Xe$

Weighted by isotopic abundance

60% is a significant difference!



Integrated Form Factor Results

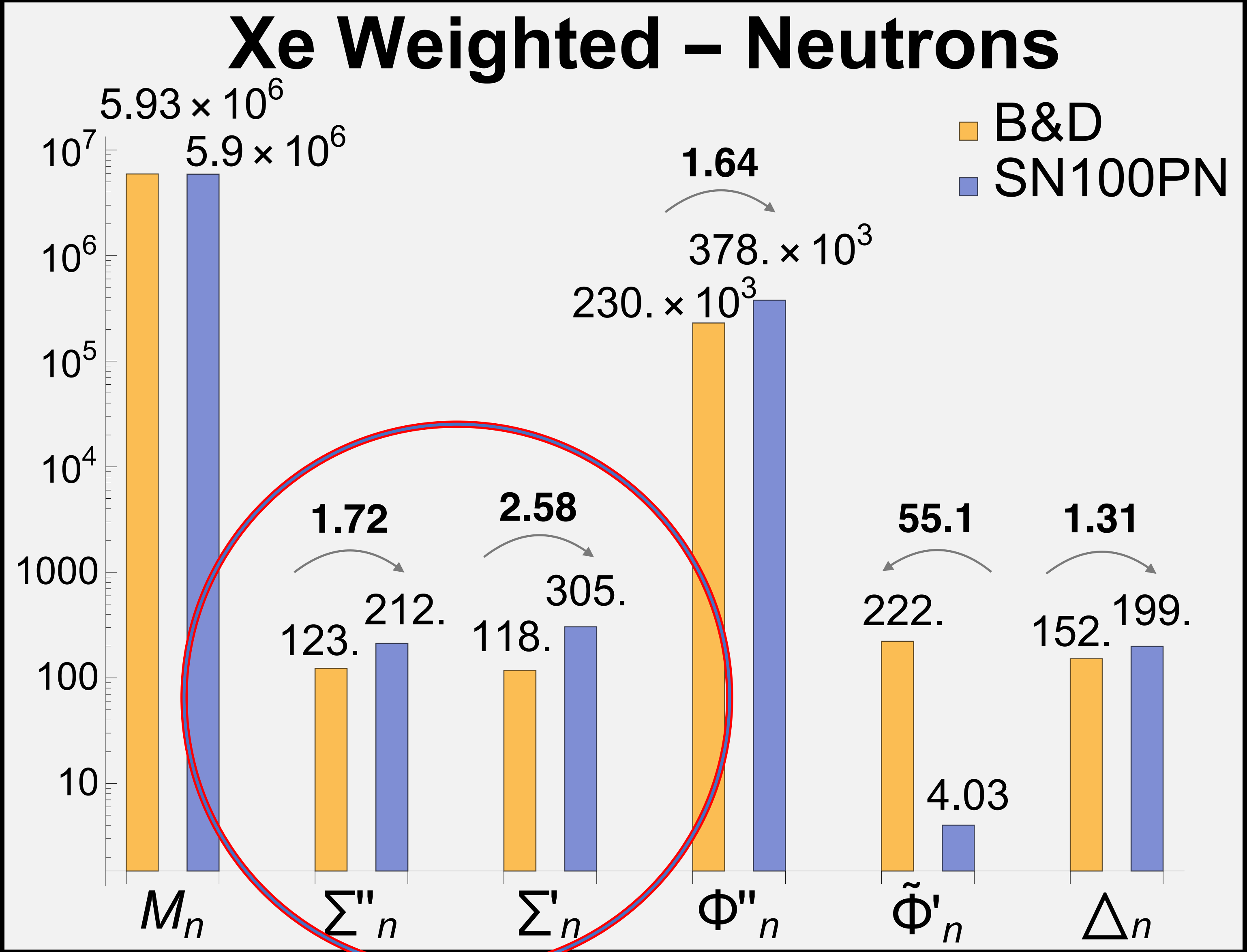
$$\int_0^{100MeV} \frac{qdq}{2} F_X^{(N,N)}(q^2)$$

$\Sigma', \Sigma'' =$ spin-dependent terms

${}^{128,129,130,131,132,134,136}Xe$

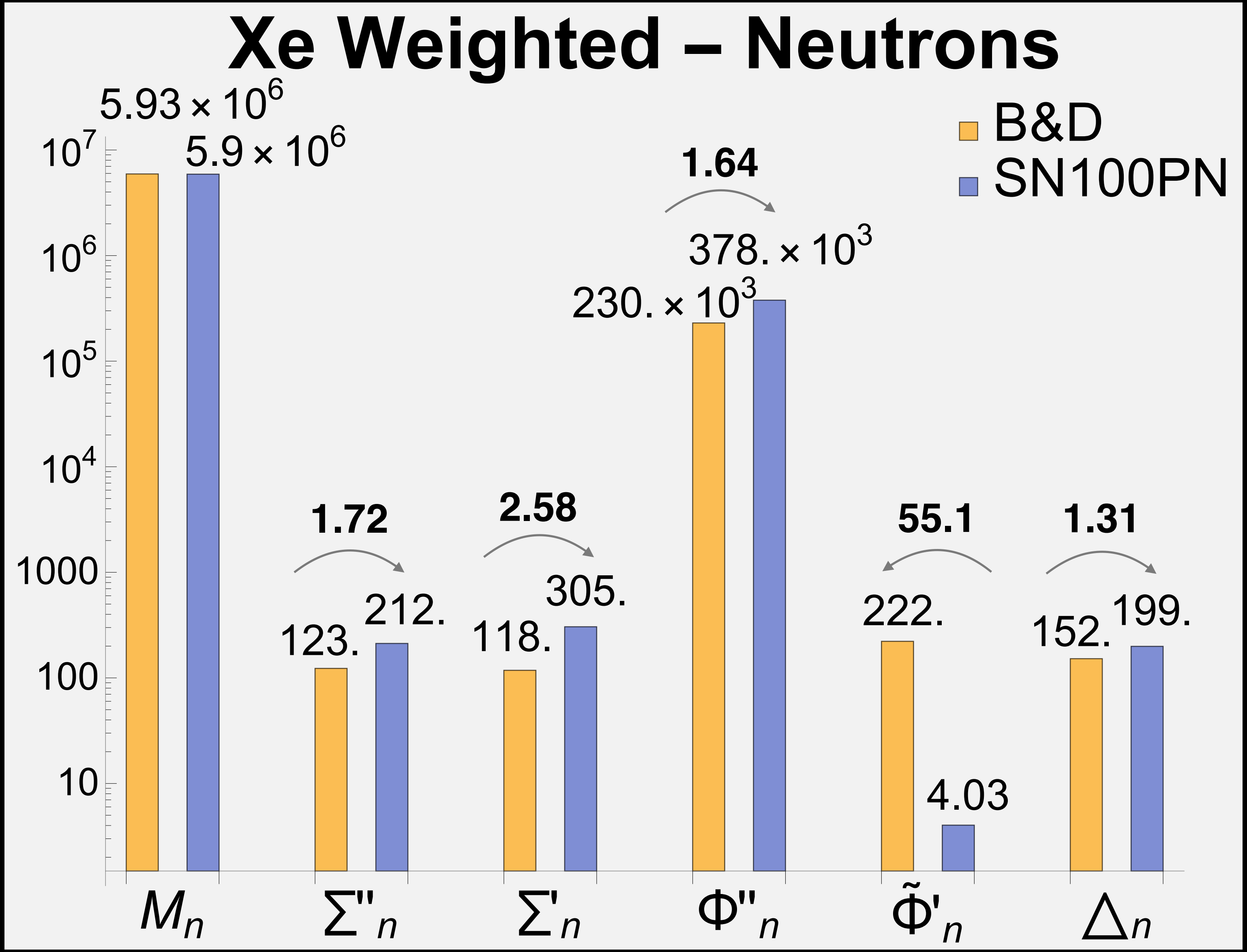
Weighted by isotopic abundance

Difference even larger for these channels



Integrated Form Factor Results

Dark Matter formalism sensitive to aspects of nuclear structure



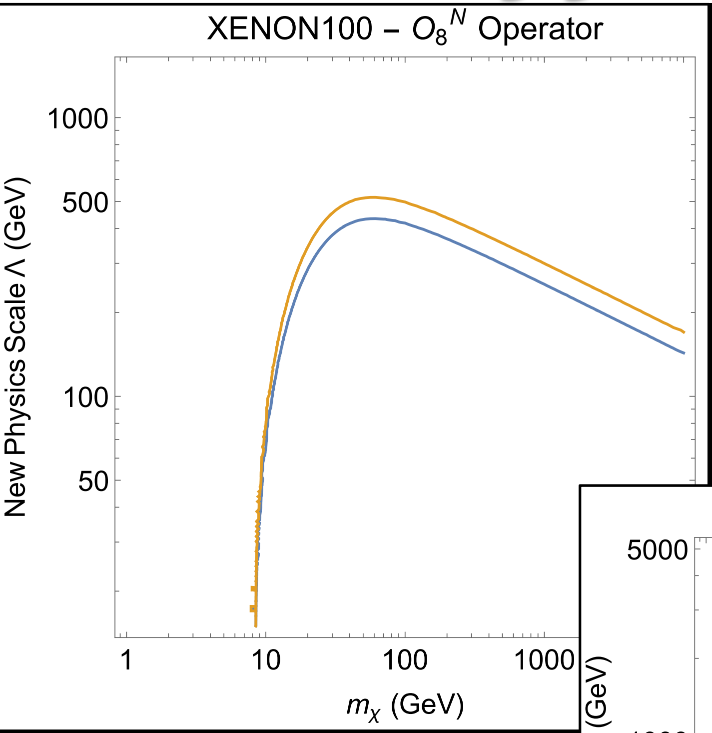
Application – LUX Experiment

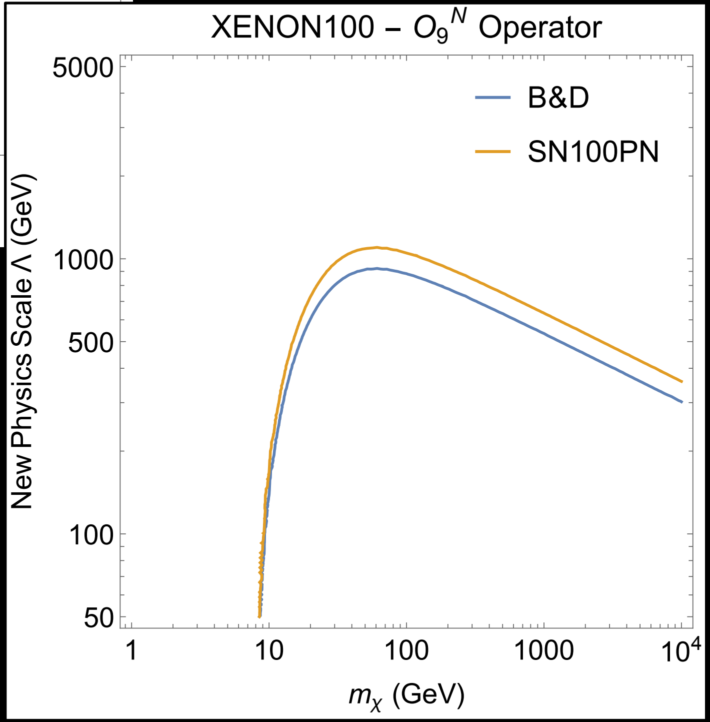
Preliminary Results

- Plot bounds on high energy physics parameter Λ
- For operators that are spin-dependent

* Methods from: M. Cirelli, E. Del Nobile, and P. Panci, JCAP 10, 019 (2013), arXiv:1307.5955 [hep-ph].

Application – LUX Experiment





Preliminary Results

Log plots

- Plot bounds on high energy physics parameter Λ
- For operators that are spin-dependent
- Λ bounds differ by 20-30%

- Cross-section

$$\sigma \propto \frac{1}{\Lambda^4}$$

* Methods from: M. Cirelli, E. Del Nobile, and P. Panci, JCAP 10, 019 (2013), arXiv:1307.5955 [hep-ph].

Important Takeaway

To constrain Dark Matter Candidates via Direct Detection

Must account for nuclear structure modelling & uncertainties!

Collaborations

Domestic:

Australian National University

Research Fellow Giorgio Busoni

- Applying nuclear form factors to various direct detection experiments/targets to calculate scattering rates

Research Fellow Jayden Newstead

- Applying nuclear form factors to neutrino elastic scattering calculations

THE UNIVERSITY OF MELBOURNE

International:

Post Doctoral Fellow Madeleine Zurowski

- Applying nuclear form factors to SuperCDMS experimental targets, Ge & Si
- 3 month research trip funded by ANU Physics travel award in 2024 to University of Toronto

SUPER CDMS

Important Takeaway

Nuclear Structure is Important!

…. Hire the nuclear physicists

Thank you for listening!

Additional Slides

Form Factors & OBDMEs

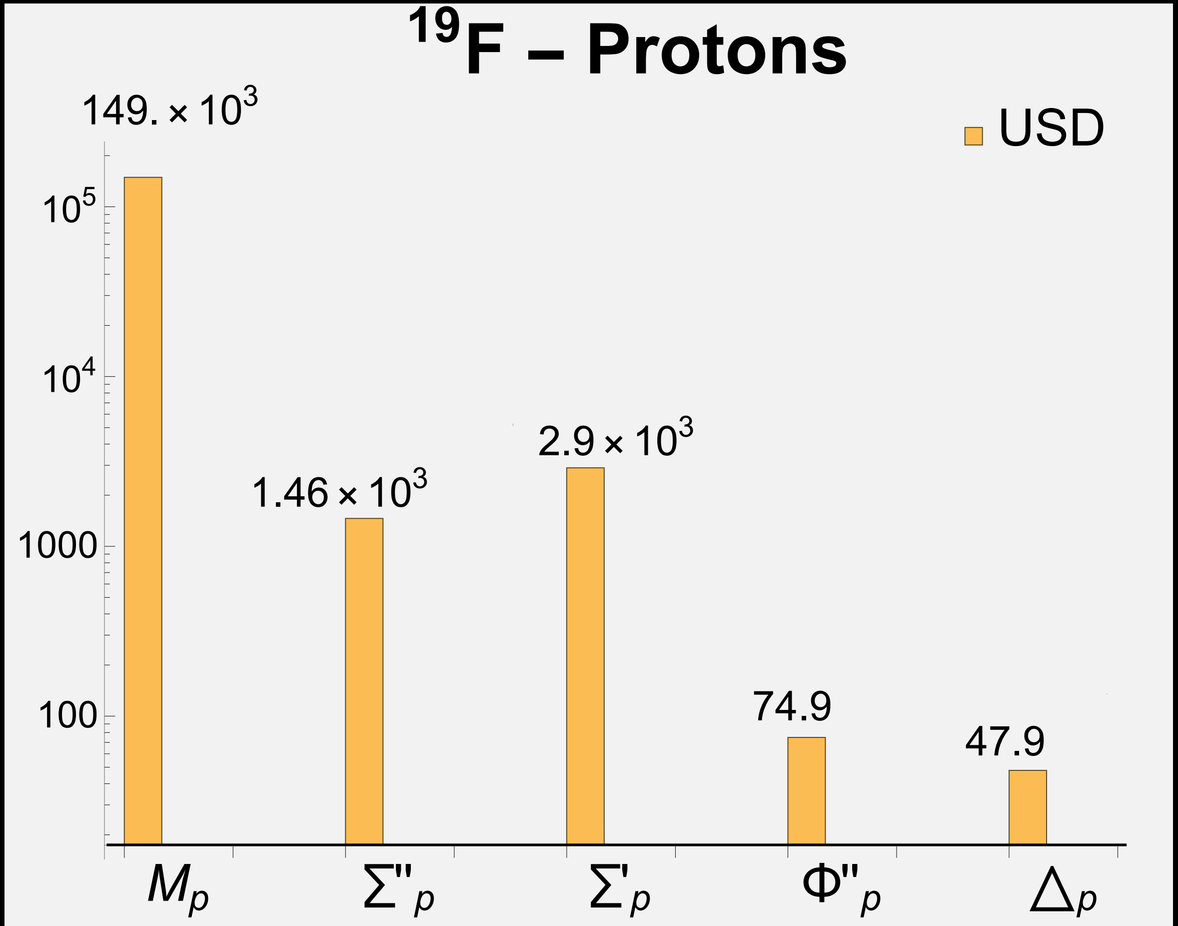
$$F^{(N,N')}_{X,Y}(q^2) \equiv \frac{4\pi}{2J_i+1} \sum_{J=0}^{2J_i} \langle J_i || X_J^{(N)} || J_i \rangle \langle J_i || Y_J^{(N')} || J_i \rangle$$

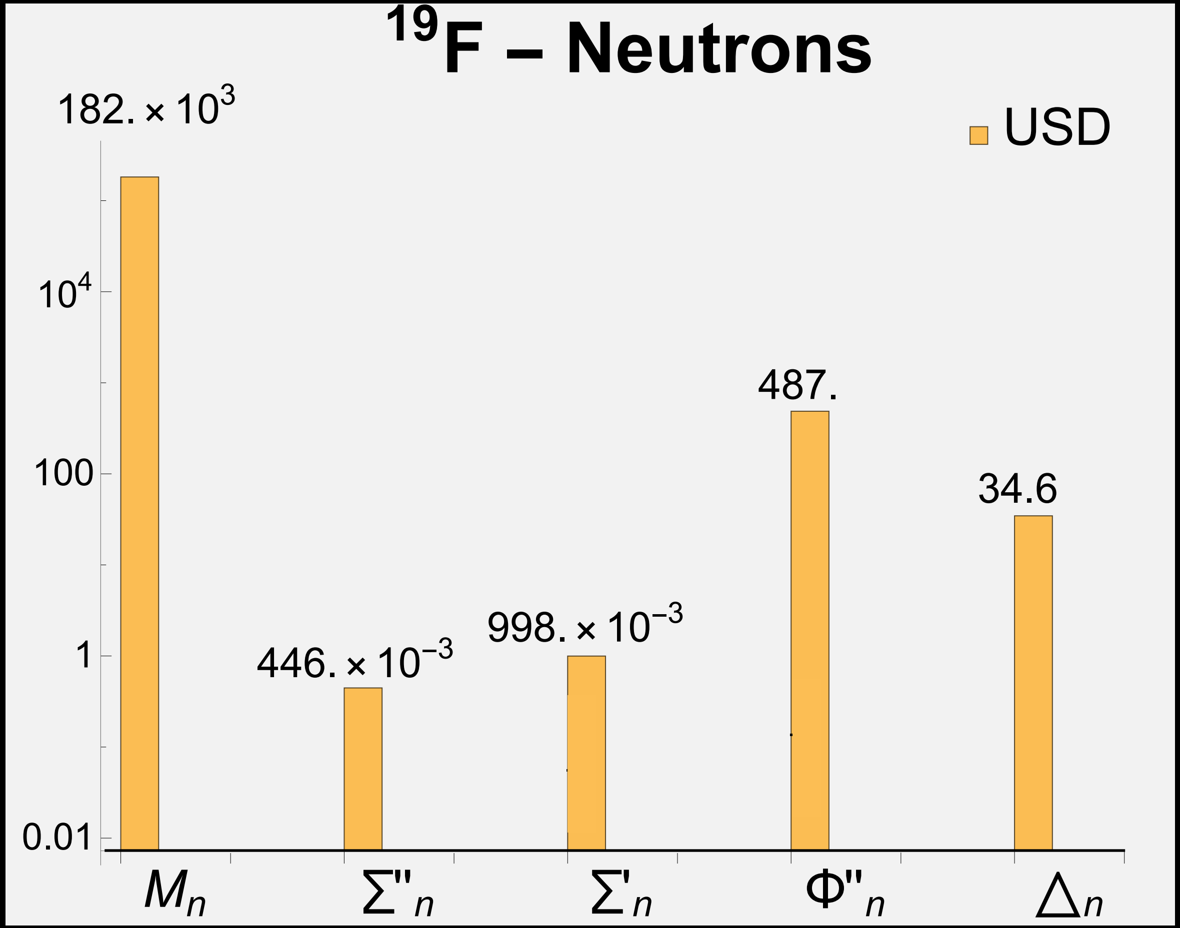
$$\begin{aligned} \langle J_i; T M_T \; || \sum_{m=1}^{A} \hat{O}_{J,\tau}(q\vec{x}_m) || \; J_i; T M_T \rangle \\ &= (-1)^{T-M_T} \begin{pmatrix} T & \tau & T \\ -M_T & 0 & M_T \end{pmatrix} \langle J_i; T \; \vdots\vdots \sum_{m=1}^{A} \hat{O}_{J,\tau}(q\vec{x}_m) \vdots\vdots \; J_i; T \rangle \\ &= (-1)^{T-M_T} \begin{pmatrix} T & \tau & T \\ -M_T & 0 & M_T \end{pmatrix} \sum_{|\alpha|,|\beta|} \Psi^{J;\tau}_{|\alpha|,|\beta|} \langle |\alpha| \; \vdots\vdots \hat{O}_{J,\tau}(q\vec{x}) \vdots\vdots \; |\beta| \rangle \end{aligned}$$

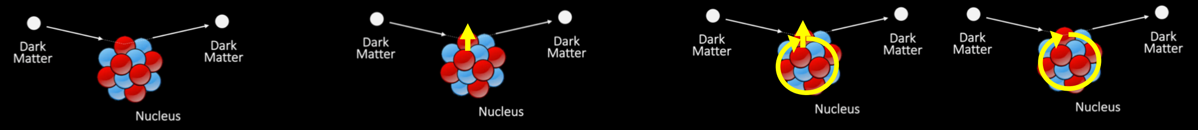
One-body density matrix elements (OBDMEs)

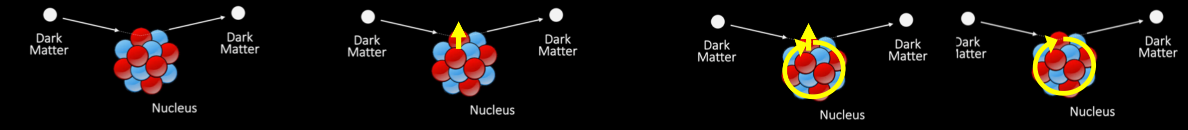
$$\Psi^{J;\tau}_{|\alpha|,|\beta|} \equiv \frac{\langle J_i; T \; \vdots\vdots \left[a^{\dagger}_{|\alpha|} \otimes \tilde{a}_{|\beta|} \right]_{J;\tau} \vdots\vdots \; J_i; T \rangle}{\sqrt{(2J+1)(2\tau+1)}}$$

Integrated Form Factor Results









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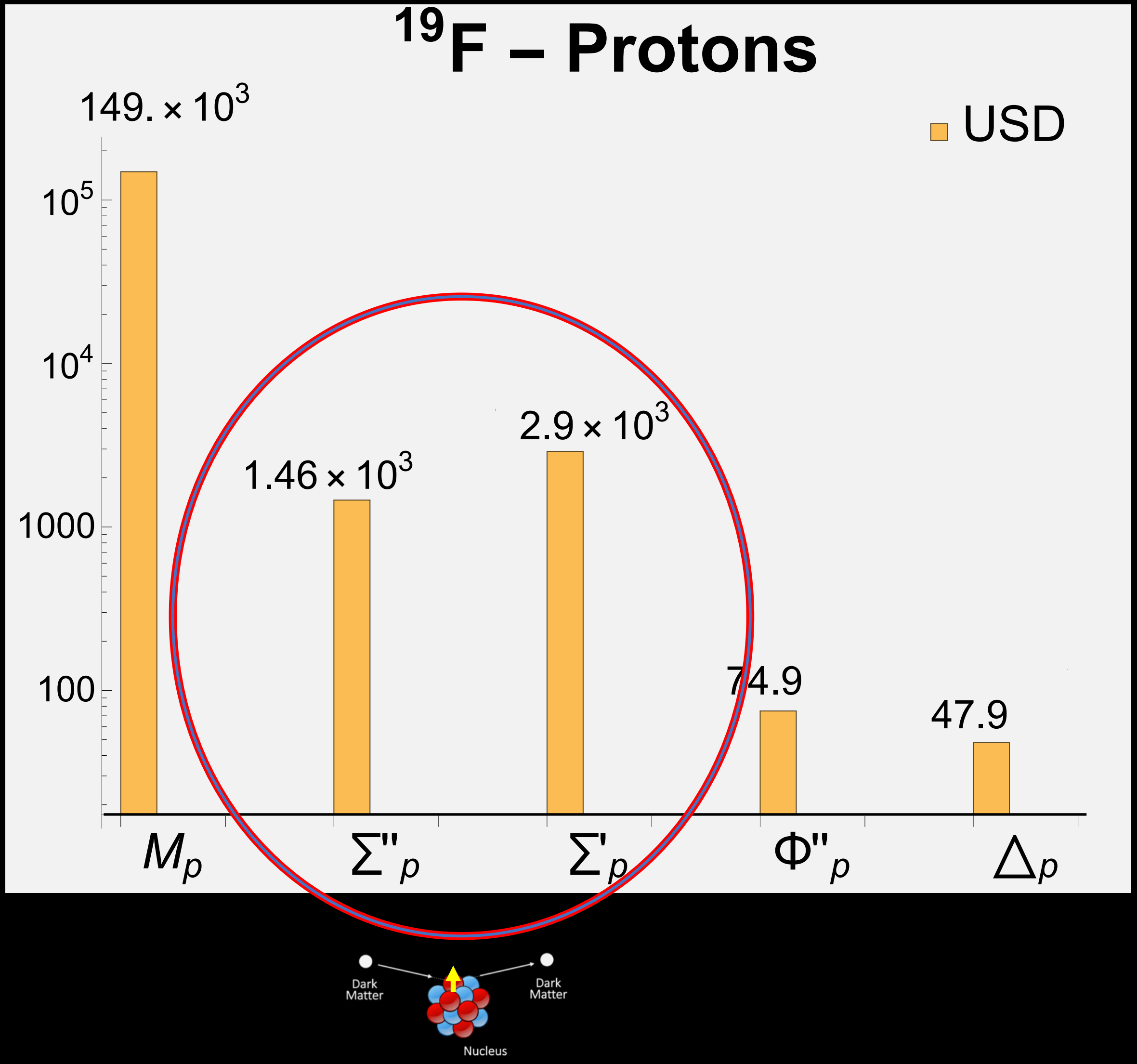

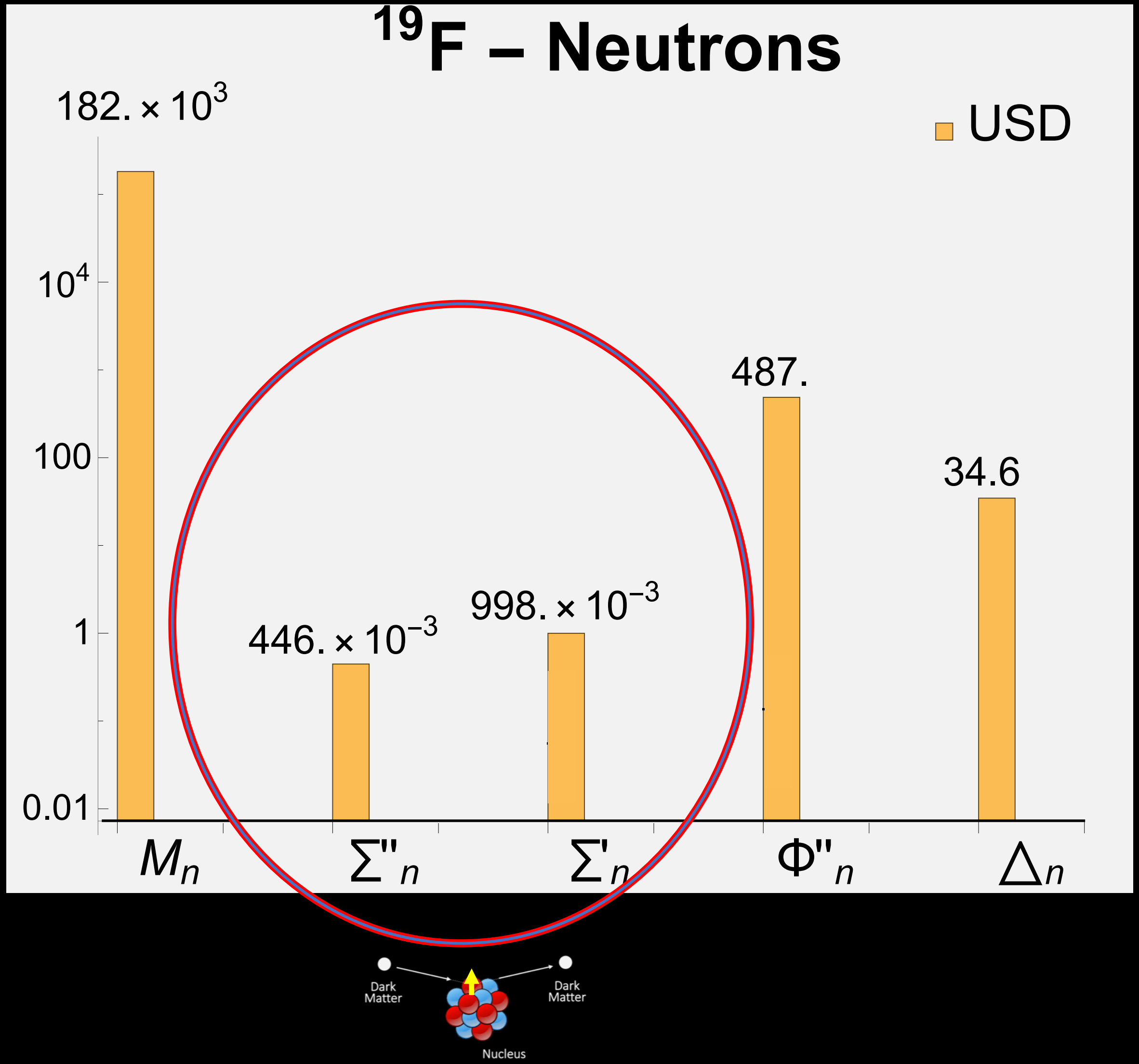

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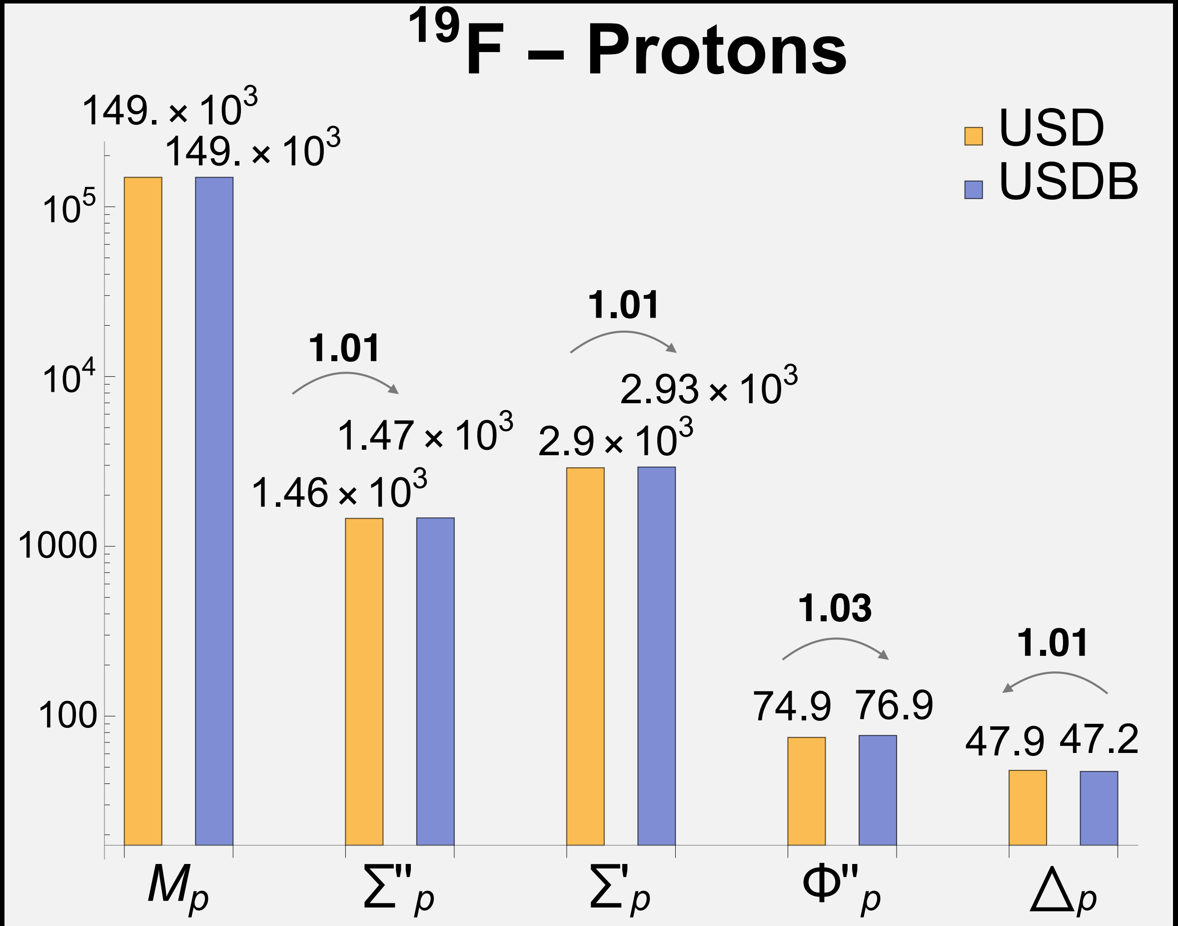

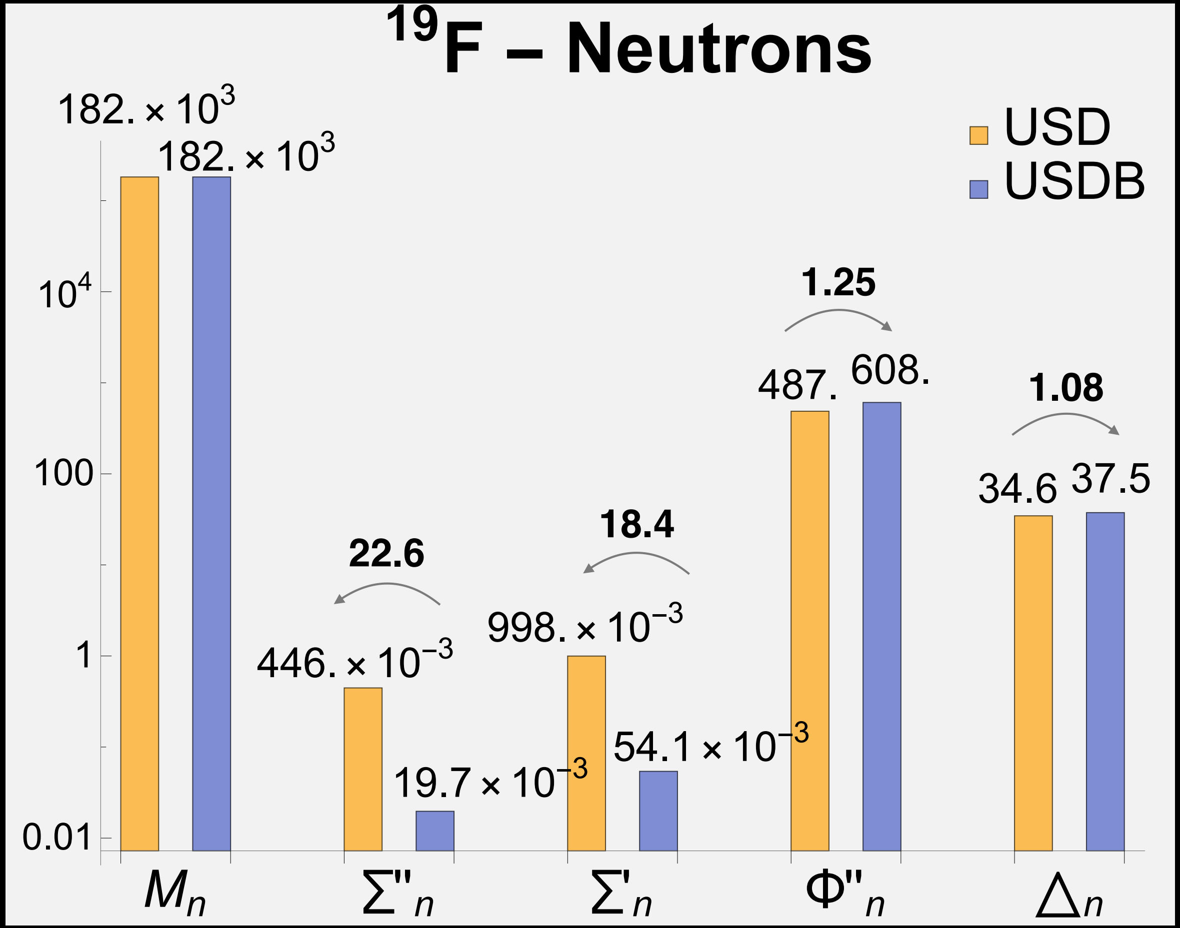

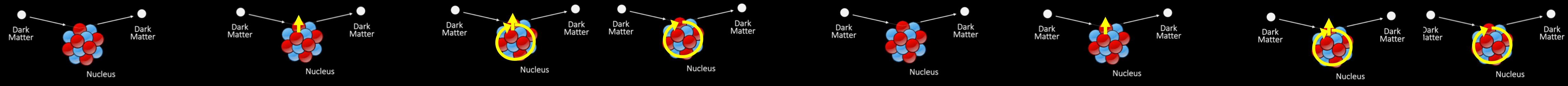

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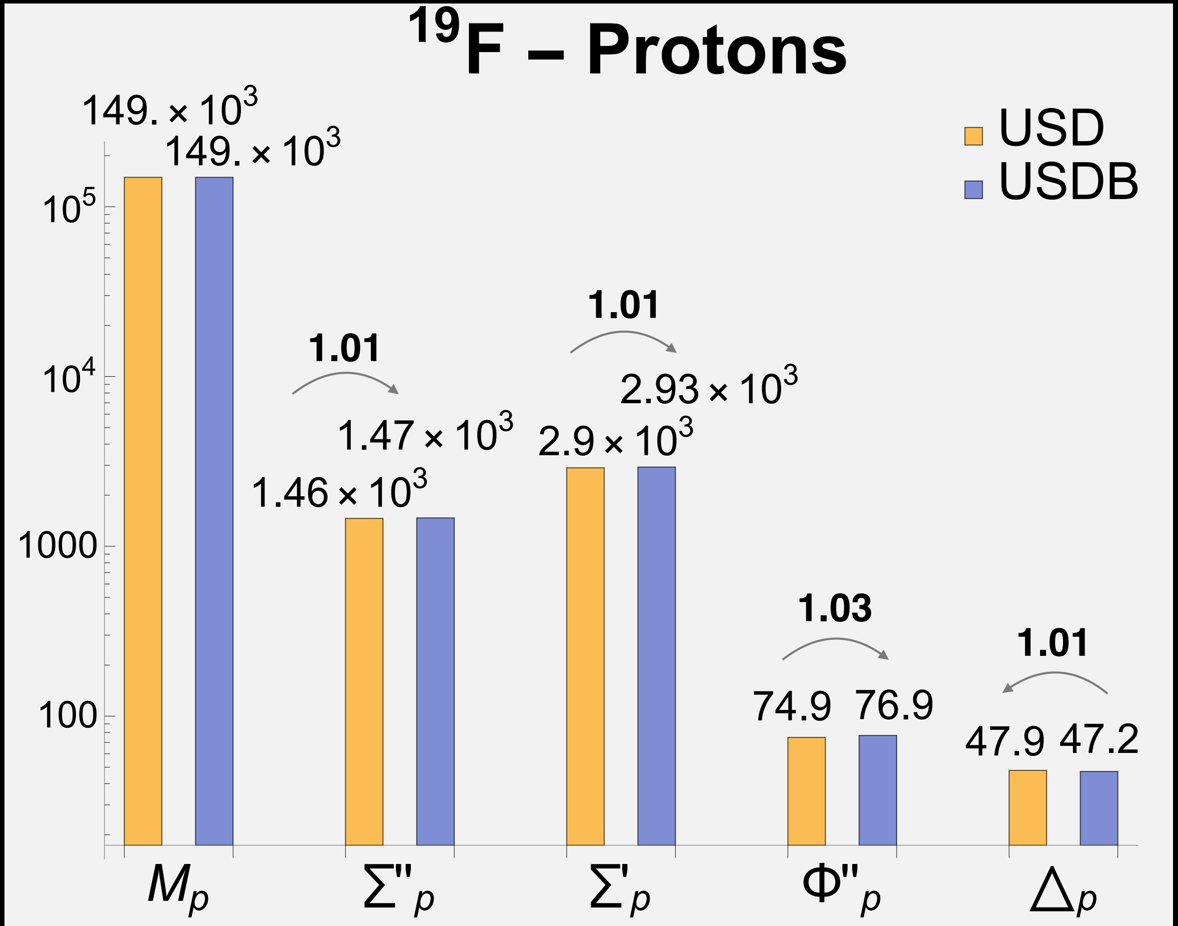
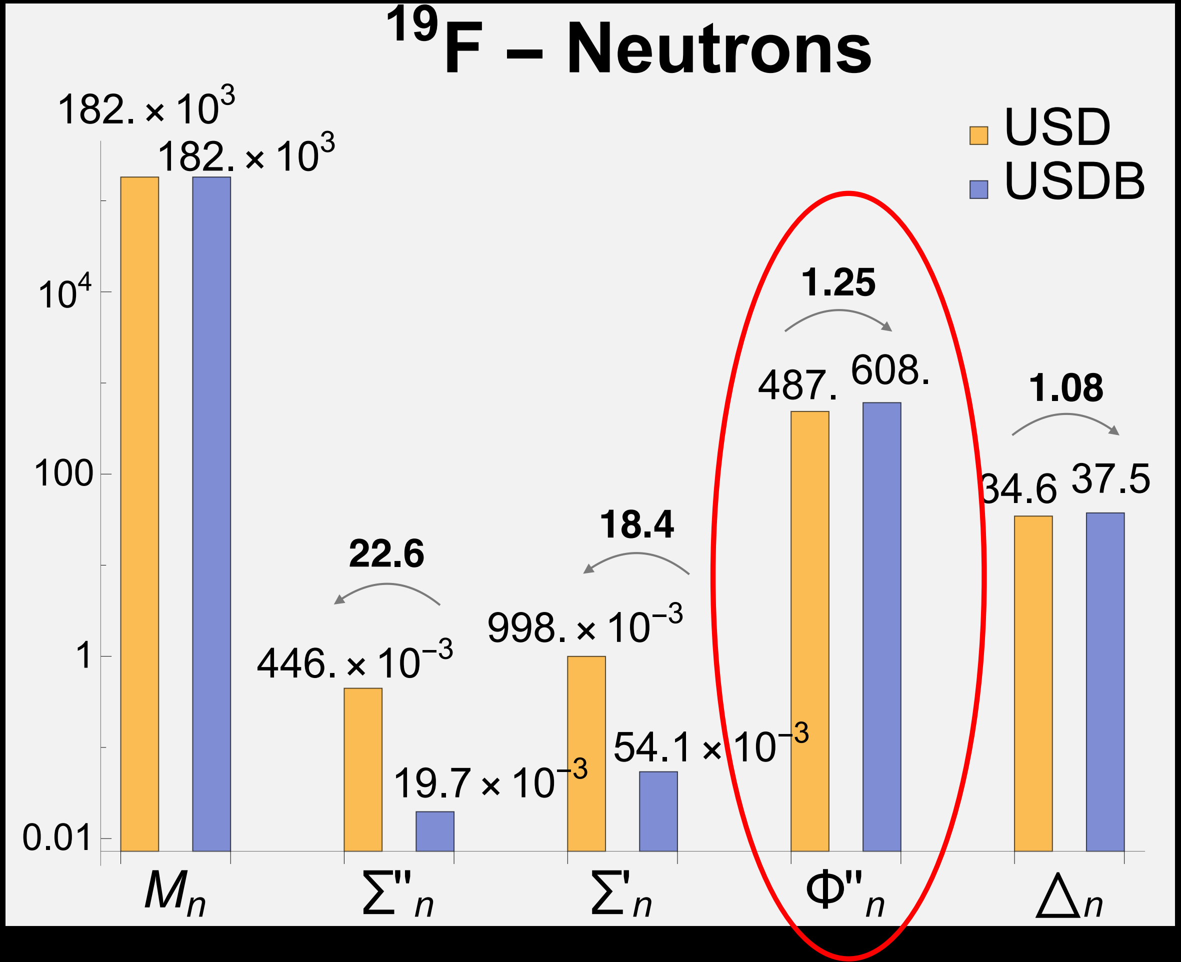
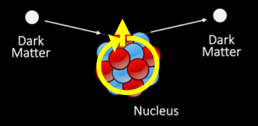
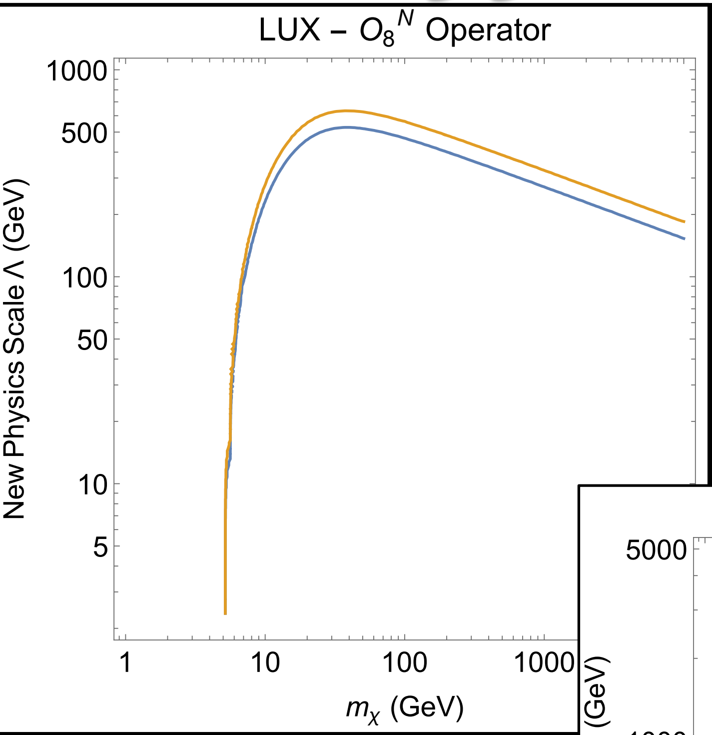


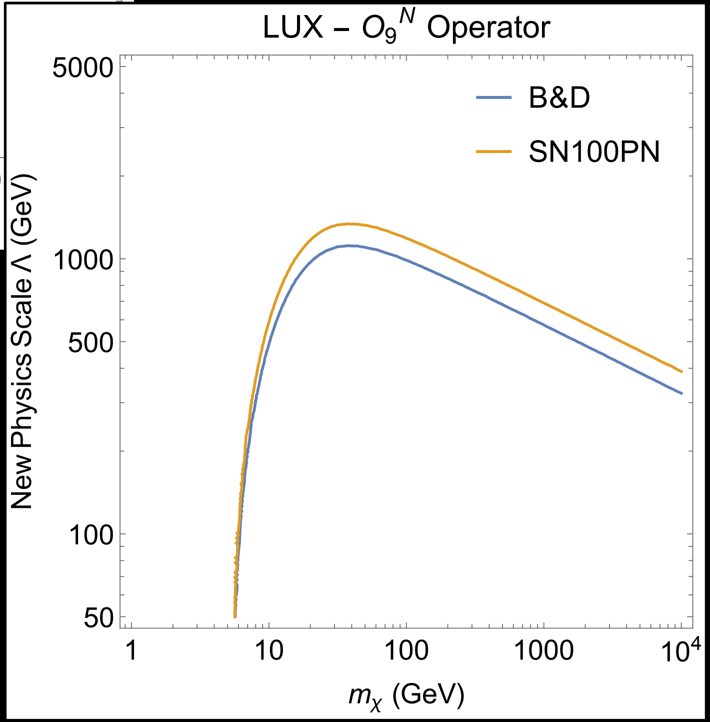


20% is a significant difference!



Application to Xenon Experiments





Preliminary Results

- Plot bounds on high energy physics parameter Λ
- For operators that are spin-dependent
- XENON100 experiment

The Setup – Nuclear Shell Model

Response $\times \left[\frac{4\pi}{2J_i+1}\right]^{-1}$	Leading Multipole	Long-wavelength Limit	Response Type
$\sum_{J=0,2,\ldots}^{\infty} \lvert\langle J_i \vert\vert M_{JM} \vert\vert J_i\rangle\rvert^2$	$M_{00}(q\vec{x}_i)$	$\frac{1}{\sqrt{4\pi}}1(i)$	M_{JM} : Charge
$\sum_{J=1,3,\ldots}^{\infty} \lvert\langle J_i \vert\vert \Sigma''_{JM} \vert\vert J_i\rangle\rvert^2$	$\Sigma''_{1M}(q\vec{x}_i)$	$\frac{1}{2\sqrt{3\pi}}\sigma_{1M}(i)$	L^5_{JM} : Axial Longitudinal
$\sum_{J=1,3,\ldots}^{\infty} \lvert\langle J_i \vert\vert \Sigma'_{JM} \vert\vert J_i\rangle\rvert^2$	$\Sigma'_{1M}(q\vec{x}_i)$	$\frac{1}{\sqrt{6\pi}}\sigma_{1M}(i)$	T^{el5}_{JM} : Axial Transverse Electric
$\sum_{J=1,3,\ldots}^{\infty} \lvert\langle J_i \vert\vert \frac{q}{m_N}\Delta_{JM} \vert\vert J_i\rangle\rvert^2$	$\frac{q}{m_N}\Delta_{1M}(q\vec{x}_i)$	$-\frac{q}{2m_N\sqrt{6\pi}}\ell_{1M}(i)$	T^{mag}_{JM} : Transverse Magnetic
$\sum_{J=0,2,\ldots}^{\infty} \lvert\langle J_i \vert\vert \frac{q}{m_N}\Phi''_{JM} \vert\vert J_i\rangle\rvert^2$	$\frac{q}{m_N}\Phi''_{00}(q\vec{x}_i)$	$-\frac{q}{3m_N\sqrt{4\pi}}\vec{\sigma}(i)\cdot\vec{\ell}(i)$	L_{JM} : Longitudinal
	$\frac{q}{m_N}\Phi''_{2M}(q\vec{x}_i)$	$-\frac{q}{m_N\sqrt{30\pi}}[x_i\otimes(\vec{\sigma}(i)\times\frac{1}{i}\vec{\nabla})_1]_{2M}$	
$\sum_{J=2,4,\ldots}^{\infty} \lvert\langle J_i \vert\vert \frac{q}{m_N}\tilde{\Phi}'_{JM} \vert\vert J_i\rangle\rvert^2$	$\frac{q}{m_N}\tilde{\Phi}'_{2M}(q\vec{x}_i)$	$-\frac{q}{m_N\sqrt{20\pi}}[x_i\otimes(\vec{\sigma}(i)\times\frac{1}{i}\vec{\nabla})_1]_{2M}$	T^{el}_{JM} : Transverse Electric

Table 1. The response DM nuclear response functions, their leading order behavior, and the response type. The notation $\otimes$ denotes a spherical tensor product, while $\times$ is the conventional cross product.

The Setup – Nuclear Shell Model

- Work directly with EFT interaction Lagrangian picture, instead of starting with a very specific high energy model, integrating out unwanted DOF, and obtaining one specific version of

$$\mathcal{L}_{int} = \chi\, \mathcal{O}_\chi\, \chi \quad N\, \mathcal{O}_N\, N$$

The Setup – Nuclear Shell Model

Constructing EFT non-relativistic interaction Hamiltonian (or Lagrangian)

Consider symmetries $\mathcal{H}_{int}$ has to obey:

Galilean invariance (shift in velocities) – need terms like

$$\vec{v} \equiv \vec{v}_{\chi,\mathrm{in}} - \vec{v}_{N,\mathrm{in}},$$

$$\vec{q} = \vec{p}' - \vec{p} = \vec{k} - \vec{k}'.$$

Hermiticity– modify above terms

$$i\vec{q}$$

$$\vec{v}^{\perp} \equiv \vec{v} + \frac{\vec{q}}{2\mu_N}.$$

The Setup – Nuclear Shell Model

Galilean, Hermitian operators in $\mathcal{H}_{int}$

$$i\vec{q}, \qquad \vec{v}^{\perp}, \qquad \vec{S}_{\chi}, \qquad \vec{S}_N.$$

- consider all combinations up to second order in $\vec{q}$, and at most quadratic in either $\vec{S}$ or $\vec{v}$.

1. P-even, S_χ-independent

$$\mathcal{O}_1 = \mathbf{1}, \qquad \mathcal{O}_2 = (v^{\perp})^2, \qquad \mathcal{O}_3 = i\vec{S}_N \cdot (\vec{q} \times \vec{v}^{\perp}),$$

2. P-even, S_χ-dependent

$$\mathcal{O}_4 = \vec{S}_\chi \cdot \vec{S}_N, \qquad \mathcal{O}_5 = i\vec{S}_\chi \cdot (\vec{q} \times \vec{v}^{\perp}), \qquad \mathcal{O}_6 = (\vec{S}_\chi \cdot \vec{q})(\vec{S}_N \cdot \vec{q}),$$

3. P-odd, S_χ-independent

$$\mathcal{O}_7 = \vec{S}_N \cdot \vec{v}^{\perp},$$

4. P-odd, S_χ-dependent

$$\mathcal{O}_8 = \vec{S}_\chi \cdot \vec{v}^{\perp}, \qquad \mathcal{O}_9 = i\vec{S}_\chi \cdot (\vec{S}_N \times \vec{q})$$

In addition, we also have T-violating operators:

5. P-odd, S_χ-independent:

$$\mathcal{O}_{10} = i\vec{S}_N \cdot \vec{q},$$

6. P-odd, S_χ-dependent

$$\mathcal{O}_{11} = i\vec{S}_\chi \cdot \vec{q}.$$

Formalism

$$\mathcal{O}_1^q = \bar{\chi}\chi\ \bar{q}q\ , \qquad \mathcal{O}_2^q = \bar{\chi}\,i\gamma^5\chi\ \bar{q}q\ ,$$
$$\mathcal{O}_3^q = \bar{\chi}\chi\ \bar{q}\,i\gamma^5 q\ , \qquad \mathcal{O}_4^q = \bar{\chi}\,i\gamma^5\chi\ \bar{q}\,i\gamma^5 q\ ,$$
$$\mathcal{O}_5^q = \bar{\chi}\gamma^\mu\chi\ \bar{q}\gamma_\mu q\ , \qquad \mathcal{O}_6^q = \bar{\chi}\gamma^\mu\gamma^5\chi\ \bar{q}\gamma_\mu q\ ,$$
$$\mathcal{O}_7^q = \bar{\chi}\gamma^\mu\chi\ \bar{q}\gamma_\mu\gamma^5 q\ , \qquad \mathcal{O}_8^q = \bar{\chi}\gamma^\mu\gamma^5\chi\ \bar{q}\gamma_\mu\gamma^5 q\ ,$$
$$\mathcal{O}_9^q = \bar{\chi}\,\sigma^{\mu\nu}\chi\ \bar{q}\,\sigma_{\mu\nu}q\ , \qquad \mathcal{O}_{10}^q = \bar{\chi}\,i\,\sigma^{\mu\nu}\gamma^5\chi\ \bar{q}\,\sigma_{\mu\nu}q\ ,$$

$$\mathcal{O}_1^N = \bar{\chi}\chi\ \bar{N}N\ , \qquad \mathcal{O}_2^N = \bar{\chi}\,i\gamma^5\chi\ \bar{N}N\ ,$$
$$\mathcal{O}_3^N = \bar{\chi}\chi\ \bar{N}\,i\gamma^5 N\ , \qquad \mathcal{O}_4^N = \bar{\chi}\,i\gamma^5\chi\ \bar{N}\,i\gamma^5 N\ ,$$
$$\mathcal{O}_5^N = \bar{\chi}\gamma^\mu\chi\ \bar{N}\gamma_\mu N\ , \qquad \mathcal{O}_6^N = \bar{\chi}\gamma^\mu\gamma^5\chi\ \bar{N}\gamma_\mu N\ ,$$
$$\mathcal{O}_7^N = \bar{\chi}\gamma^\mu\chi\ \bar{N}\gamma_\mu\gamma^5 N\ , \qquad \mathcal{O}_8^N = \bar{\chi}\gamma^\mu\gamma^5\chi\ \bar{N}\gamma_\mu\gamma^5 N\ ,$$
$$\mathcal{O}_9^N = \bar{\chi}\,\sigma^{\mu\nu}\chi\ \bar{N}\,\sigma_{\mu\nu}N\ , \qquad \mathcal{O}_{10}^N = \bar{\chi}\,i\,\sigma^{\mu\nu}\gamma^5\chi\ \bar{N}\,\sigma_{\mu\nu}N\ ,$$

$$c_{1,2}^N = \sum_{q=u,d,s} c_{1,2}^q \frac{m_N}{m_q} f_{Tq}^{(N)} + \frac{2}{27} f_{TG}^{(N)} \left(\sum_{q=c,b,t} c_{1,2}^q \frac{m_N}{m_q} - c_{1,2}^g m_N \right)\ ,$$
$$c_{3,4}^N = \sum_{q=u,d,s} \frac{m_N}{m_q} \left[(c_{3,4}^q - C_{3,4}) + c_{3,4}^g \bar{m} \right] \Delta_q^{(N)}\ ,$$
$$c_{5,6}^p = 2\,c_{5,6}^u + c_{5,6}^d\ , \qquad c_{5,6}^n = c_{5,6}^u + 2\,c_{5,6}^d\ ,$$
$$c_{7,8}^N = \sum_q c_{7,8}^q\,\Delta_q^{(N)}\ ,$$
$$c_{9,10}^N = \sum_q c_{9,10}^q\,\delta_q^{(N)}\ ,$$

Formalism

$$
\begin{aligned}
\langle \mathcal{O}_1^N \rangle = \langle \mathcal{O}_5^N \rangle &= 4 m_\chi m_N \mathcal{O}_1^{\mathrm{NR}} \ , \\
\langle \mathcal{O}_2^N \rangle &= -4 m_N \mathcal{O}_{11}^{\mathrm{NR}} \ , \\
\langle \mathcal{O}_3^N \rangle &= 4 m_\chi \mathcal{O}_{10}^{\mathrm{NR}} \ , \\
\langle \mathcal{O}_4^N \rangle &= 4 \mathcal{O}_6^{\mathrm{NR}} \ , \\
\langle \mathcal{O}_6^N \rangle &= 8 m_\chi \left(+ m_N \mathcal{O}_8^{\mathrm{NR}} + \mathcal{O}_9^{\mathrm{NR}} \right) \ , \\
\langle \mathcal{O}_7^N \rangle &= 8 m_N \left(- m_\chi \mathcal{O}_7^{\mathrm{NR}} + \mathcal{O}_9^{\mathrm{NR}} \right) \ , \\
\langle \mathcal{O}_8^N \rangle = -\frac{1}{2} \langle \mathcal{O}_9^N \rangle &= -16\, m_\chi m_N \mathcal{O}_4^{\mathrm{NR}} \ , \\
\langle \mathcal{O}_{10}^N \rangle &= 8 \left(m_\chi \mathcal{O}_{11}^{\mathrm{NR}} - m_N \mathcal{O}_{10}^{\mathrm{NR}} - 4 m_\chi m_N \mathcal{O}_{12}^{\mathrm{NR}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{O}_1^{\mathrm{NR}} = \mathbb{1} \ , && \\
&\mathcal{O}_3^{\mathrm{NR}} = i\, \vec{s}_N \cdot (\vec{q} \times \vec{v}^\perp) \ , && \mathcal{O}_4^{\mathrm{NR}} = \vec{s}_\chi \cdot \vec{s}_N \ , \\
&\mathcal{O}_5^{\mathrm{NR}} = i\, \vec{s}_\chi \cdot (\vec{q} \times \vec{v}^\perp) \ , && \mathcal{O}_6^{\mathrm{NR}} = (\vec{s}_\chi \cdot \vec{q})(\vec{s}_N \cdot \vec{q}) \ , \\
&\mathcal{O}_7^{\mathrm{NR}} = \vec{s}_N \cdot \vec{v}^\perp \ , && \mathcal{O}_8^{\mathrm{NR}} = \vec{s}_\chi \cdot \vec{v}^\perp \ , \\
&\mathcal{O}_9^{\mathrm{NR}} = i\, \vec{s}_\chi \cdot (\vec{s}_N \times \vec{q}) \ , && \mathcal{O}_{10}^{\mathrm{NR}} = i\, \vec{s}_N \cdot \vec{q} \ , \\
&\mathcal{O}_{11}^{\mathrm{NR}} = i\, \vec{s}_\chi \cdot \vec{q} \ , && \mathcal{O}_{12}^{\mathrm{NR}} = \vec{v}^\perp \cdot (\vec{s}_\chi \times \vec{s}_N) \ ,
\end{aligned}
$$

Likelihood Ratio Test Statistic

$$\mathcal{L}(\vec{N}^{\mathrm{obs}} \,|\, \lambda) = \prod_k \frac{N_k(\lambda, m_\chi)^{N_k^{\mathrm{obs}}}}{N_k^{\mathrm{obs}}!} \, e^{-N_k(\lambda, m_\chi)}$$

$$\mathrm{TS}(\lambda, m_\chi) = -2 \ln \left(\mathcal{L}(\vec{N}^{\mathrm{obs}} \,|\, \lambda) / \mathcal{L}_{\mathrm{bkg}} \right)$$

$$\mathrm{TS}(\lambda, m_\chi) = -2 \sum_k N_k^{\mathrm{obs}} \ln \left(\frac{N_k^{\mathrm{th}}(\lambda, m_\chi) + N_k^{\mathrm{bkg}}}{N_k^{\mathrm{bkg}}} \right) + 2 \sum_k N_k^{\mathrm{th}}(\lambda, m_\chi)$$